Where does the formula come from?

Students investigating total surface areas of a pyramid and cone using models and technology

Samuel Obara
Texas State University, USA
<so16@txstate.edu>

Spatial reasoning is a skill that needs to be developed in students as it is important in geometry for determining total surface areas and volumes of 3-dimensional shapes (Liedtke, 1995). Simply teaching children the formulae, in this case for finding total surface areas, can limit them in understanding mathematics conceptually (Bonotto, 2003). As pointed out in the NCTM *Principles and Standards* (2000):

> Some students may have difficulty finding the surface area of three-dimensional shapes using two-dimensional representations because they cannot visualize the unseen faces of the shapes. Experience with models of three-dimensional shapes and their two-dimensional "nets"[1] is useful in such visualization… Students should build three-dimensional objects from two-dimensional representations (p. 237).

1. Ainge, (1996) defines a net as "a plane diagram showing all faces of a 3D shape, which can be cut out and folded to construct the solid" (p. 346).

Helping students establish a relationship between the total surface area of a three-dimensional solid and the area of a two-dimensional net should help them in understanding total surface area conceptually. In this article the challenges pre-service secondary school teachers faced in trying to understand how to work out the surface areas of square and rectangular pyramids and a cone are discussed. The pitfalls of just giving students a formula to work with without involving them in learning where the formulas come from are presented.

Investigating the surface area of a pyramid using its visual representation

I started the activity by providing my students with a right square pyramid of base side length *s*, and height *h* (Figure 1a).[2] Students were asked to come up with strategies for finding the total surface area. One group of students simply googled the formula for finding the total surface area of a square pyramid and found it to be

$$A = s\left(s + \sqrt{s^2 + 4h^2}\right)$$

2. Note that the model comes with its net inside.

Asked to explain what the formula meant, the students could not resolve why the formula was in that form, but were convinced that it would work in finding the total surface area. The students were even more confused when asked to explain $\sqrt{s^2 + 4h^2}$ in the formula. The group of students started to investigate what the formula meant and why it worked. One student asked the rest to think about what it meant to find the total surface area of the square pyramid. In response, another student answered that it meant "find the area of all the faces and adding them all." Visual representation of the solid helped the students visualise its base and lateral faces. The students started counting the number and type of faces in the figure and asking each other how to find individual areas in the square pyramid. Another student asked, "What about if we find the area of the base and add it to the four triangular faces?" The students seemed to agree about the strategy of finding the total surface area but were not sure if that would result in

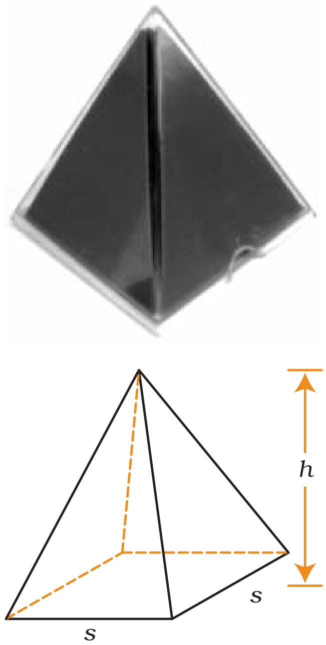

Figure 1. Square pyramid.

$$A = s\left(s + \sqrt{s^2 + 4h^2}\right)$$

(where A represents total surface area). They started finding the area of the base $s \times s = s^2$ and discussed how to find the area of the four triangular faces. The students knew that the formula for finding the area of a triangle is

$$\frac{1}{2}\text{base} \times \text{ perpendicular height}$$

Then, using the model (Figure 1) the students noted that the pyramid had height h and base length s. Holding the model and thinking, one student pointed out that since the base length is s and the height of the pyramid is h, then the total area of the four triangles would be

$$4\left(\frac{1}{2}s \times h\right)$$

Some students tended to agree with that approach, but others were not sure.

The students were struggling with understanding the formula

$$4\left(\frac{1}{2}s \times h\right)$$

One student asked, "What does this formula really mean?" Holding the model (Figure 1), another student pointed out how can they might use the height of the pyramid to find the areas of the four triangles. At that point, the students realised that to find the area of the triangular faces, what they really needed was the height of the triangular face (the slant height of the pyramid, l, and not the height of the pyramid, h). Without models in hand, some students could not see the perpendicular relationship between EF and CD in lateral triangle ECD (Figure 2). However, the model (Figure 1) helped in that visualisation.

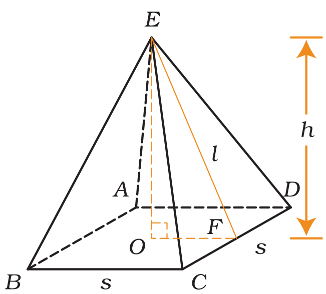

Figure 2. Square pyramid with height l of triangular faces.

The students were then faced with finding the length l (height of triangular face, Figure 2), but struggled with how to put it in terms of s and h as found in the formula. One student suggested employing the Pythagorean theorem by using one of the slant edges of the pyramid EC or ED as the

hypotenuse and $\frac{1}{2}s = CF = FD$ as one of the legs of the right triangle. This strategy did not work because the slant height l needed to be expressed in terms of h, the height of the pyramid. Another student suggested dropping a line perpendicular to the base, $ABCD$, from point E to meet at point O (Figure 2). This was difficult for students to visualise but utilising the model made it easier to see the height EO. The student demonstrated that using the right triangle (EOF) with its legs EO = height (h) of the pyramid and $OF = \frac{1}{2}s$, the length l (hypotenuse) could be calculated. Using that reasoning, the students found l as follows:

$$l = \sqrt{\left(\left(\frac{1}{2}s\right)^2 + h^2\right)}$$
$$= \sqrt{\left(\frac{s^2}{4} + h^2\right)}$$

Therefore the total surface area is

$$A = \text{area of the base} + \text{area of the lateral surfaces}$$
$$= s^2 + 4 \times \frac{1}{2}s\sqrt{\left(\frac{s^2}{4} + h^2\right)}$$
$$= s^2 + 2s\sqrt{\left(\frac{s^2 + 4h^2}{4}\right)}$$
$$= s\left(s + \sqrt{s^2 + 4h^2}\right)$$

The students were still not able to comfortably make sense of the formula. In addition, it remained difficult for them to generalise the formula for other pyramids that do not have a square base.

Using physical models and their associated nets

A second group of students employed a different strategy. The students took the net out of the model and opened it (Figure 3b) and then folded it back up and replaced it in the pyramid. I noted students doing this over and over again, moving to two-dimensional representation of a three-dimensional solid in an effort to make sense of the total surface area of the pyramid. As the students continued with investigation, they noted that sometimes it was hard to visualise the total surface area of the solid but that it is made much easier when a net accompanied the solid (Figure 3).

In this case, the students labelled the area of each of the four isosceles triangles with height l and a square with sides s, and then started finding the individual areas. They began with the lateral area (the four isosceles triangles) by drawing a diagram using the dynamic software *Geometer's Sketchpad* (Jackiw, 2001) to illustrate the net as shown in Figure 4.

Figure 3. Net for the square pyramid.

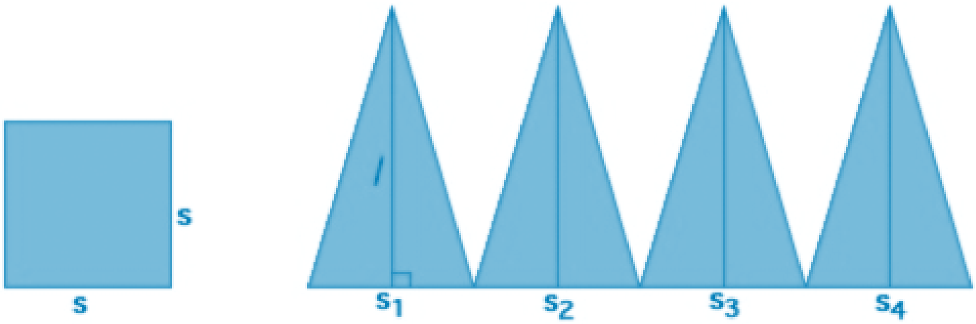

Figure 4. Square base and four triangular faces.

The students noted that:

$$\text{Area of a square} = s^2$$

$$\text{Area of each triangle} = \frac{1}{2} s \times l$$

$$\begin{aligned}\text{Total surface area} &= \text{area of the base} + \text{lateral surface area}\\ &= s^2 + \frac{1}{2} s_1 \times l + \frac{1}{2} s_2 \times l + \frac{1}{2} s_3 \times l + \frac{1}{2} s_4 \times l\\ &= \frac{1}{2} l(s_1 + s_2 + s_3 + s_4)\\ &= \text{area of the base} + \frac{1}{2} l \times (\text{perimeter of base})\end{aligned}$$

Using *Geometer's Sketchpad*, students also investigated the formula derived by finding the area as shown in Figure 5. Due to the dynamic nature of *Geometer's Sketchpad*, the students could change the base length measurement, note the change in the sum of all the areas and that the area formula derived yielded the same results as shown (Figure 5).

I then asked my students what they thought would happen if the base had a regular octagonal base. They came up with this formula for the total surface area:

Total surface area of octagonal pyramid (TSA)

$$= \text{area of octagon} + \text{lateral surface area}$$

$$= \text{Area of octagon} + \frac{1}{2} s_1 \times l + \frac{1}{2} s_2 \times l + \frac{1}{2} s_3 \times l + \frac{1}{2} s_4 \times l + \frac{1}{2} s_5 \times l + \frac{1}{2} s_6 \times l + \frac{1}{2} s_7 \times l + \frac{1}{2} s_8 \times l$$

$$= \text{Area of octagon} + \frac{1}{2} l(s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8)$$

$$= \text{Area of the base} + \frac{1}{2} l \times (\text{perimeter of the base})$$

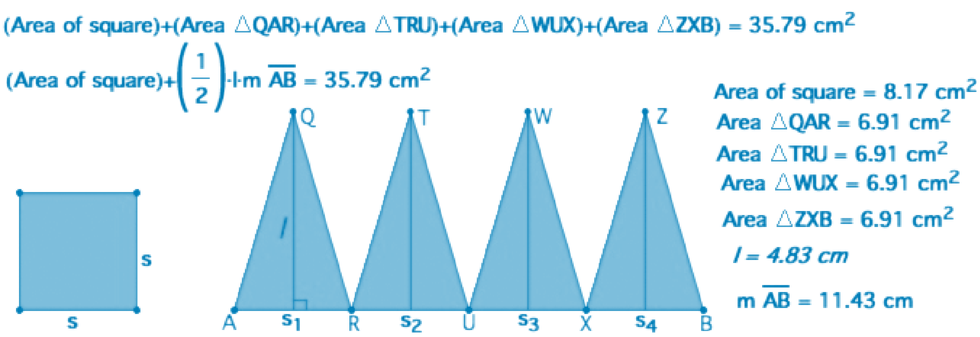

Figure 5. Area investigation of a square pyramid.

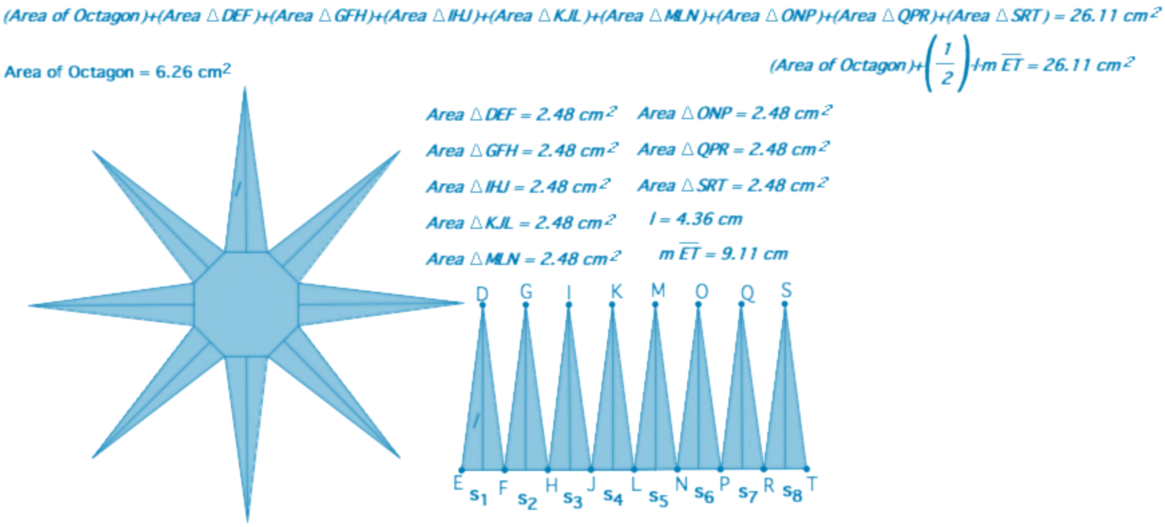

Figure 6. Area investigation of an octagonal pyramid.

Using a similar approach and also investigating the same formula, *Geometer's Sketchpad* was used to construct a net and measurements were recorded as shown in Figure 6.

I then challenged my students to think about what would happen if the base of the polygon had 100 sides? What about 1000 sides? What about 10 000 sides and what would the lateral faces look like? The students responded that as the number of sides of the polygon increases, the polygon approaches a circle and the solid becomes a cone. I then asked them to use the same approach to find the total surface area of a cone and they suggested:

$$\begin{aligned}\text{Total surface area of a cone} &= \text{area of a circle} + \text{lateral surface area}\\ &= \pi r^2 + \frac{1}{2} l \times (\text{perimeter of base})\\ &= \pi r^2 + \frac{1}{2} l \times (2\pi r)\\ &= \pi r^2 + \pi r l \quad (\text{where } r \text{ is the radius of a circle})\end{aligned}$$

During classroom reflection of the activity, some students noted in their notebook that they never had any idea how the formula of finding the lateral surface area of a cone (πrl) was derived before this experience. It was a great learning experience for them as one student stated:

> The use of nets in this activity has greatly helped my understanding of total surface area. Using nets enables me to see all the faces at a glance and for a person like me who is not good at visualisation, having models and nets greatly help. In the past, teachers did hand me a formula to use and I never questioned how the formula was derived. I love looking at more than one way of deriving the formula.

Constructing the net of a cone with GSP

In order to further develop students' three-dimensional visualisation, particularly their proportional reasoning abilities, I involved students in constructing the net of a cone. To create an extension activity on the total surface area of a cone, I gave this group of students a model of a cone (Figure 7a, b) that came with its net (Figure 7c). The students demonstrated how the net enabled them to move from a two-dimensional representation to a three-dimensional solid.

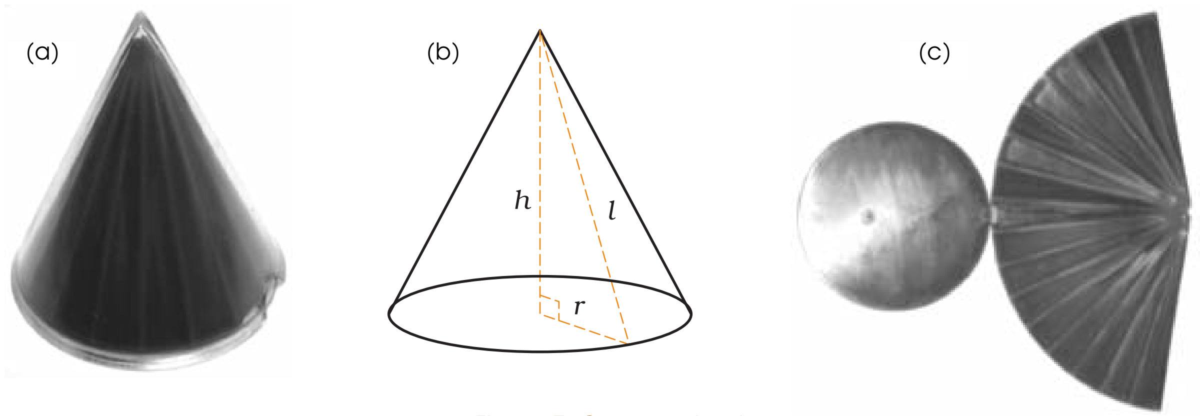

Figure 7. Cone and net.

The students were asked to investigate how the net was constructed. One student noted that the base was easy to construct as long as the radius was known. The lateral surface proved to be challenging. As the students engaged in class discussion and reflection, one student remarked that the lateral surface was a sector of another circle with radius equal to the slant height l, of the cone (note that $l = \sqrt{r^2 + h^2}$) and that they could use l to construct a circle with a sector equivalent to the lateral surface of the cone. However, the challenge was how to construct the sector, in other words, how to construct the net of a cone of radius r and height h. The students decided to use *Geometer's Sketchpad* to construct the net. They listed the information they had to work with in the construction process:

- We know the length of the arc (arc length of the net) is the circumference of circular base of the cone.
- We know the radius of the base of the cone.
- We know the height of the cone, h, and its slant height, l.

Holding the net, folding it to make a cone, and then looking back to the net helped students visualise the construction. One student suggested that they needed to construct the circle of radius l and then construct the arc on it but was not sure how to go about it. The students got frustrated and needed help. I suggested that they consider the following:

- Note that the circumference C of a circle of diameter D is an arc length, which means that the radian measure
$$\pi = \frac{C}{D};$$
since $D = 2r$ (r = the radius of a circle), then
$$\pi = \frac{C}{2r} \Rightarrow 2\pi = \frac{C}{r} = \text{radian measure}$$
- With that said, I noted to the students that the ratio of any arc length, s, to the radius, r, that is,
$$\left(\frac{\text{arc length}}{\text{radius}} = \frac{s}{r}\right)$$
is actually the radian measure of the central angle that is subtended by the arc.
- the Radian measure $\theta = \frac{s}{r}$ (see Figure 8).

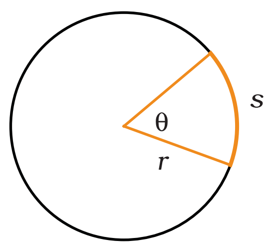

Figure 8. Radian measure.

I then let students grapple with the net construction and, as a group, they came up with the following steps:

- Start with a radius of length *AB*, and then construct circle c_1. Then through point *B* (the centre of the circle), construct a line perpendicular to radius *AB* to some point *C* outside the circle to form *BC* the height of the cone.
- Using points *A* and *C*, construct the slant height of the cone.
- Using point *D* as the center and *AC* as the radius, construct circle c_2 (Figure 9).
- Measure the circumference of c_1 and the slant height *AC*.
- Calculate ratio
  $$\frac{\text{circumference } c_1}{\text{CA}}$$
- Construct point *E* on c_2 and construct radius *ED*.
- Double click on point *D* and then select radius *ED* and point *E*. Under transformation menu, select rotate through
  $$\frac{\text{circumference } c_1}{\text{CA}}$$
  to construct point *E*′.
- Select point *E* followed by the circle (c_2) and then *E*′ to construct arc *EE*′.

I then asked the students to investigate circle c_2 in relationship to sector *EE*′. Using GSP, some students investigated (Figure 10) the relationship between:

- the arc length *EE*′ and the chord length *E*′*E*.
- the arc length *EE*′ and circumference c_2.
- area of sector *EE*′ and area of circle c_2.

The main goal here was to invoke the students' knowledge of proportional reasoning as the NCTM *Principles and Standards* (2000) suggest "students should become proficient in creating ratios to make comparisons in situations that involve pairs of numbers" (p. 34). The students pointed out that the ratio

$$\frac{\text{arc length } EE'}{\text{circumference } c_2}$$

and

$$\frac{\text{area of sector}}{\text{area of circle } c_2}$$

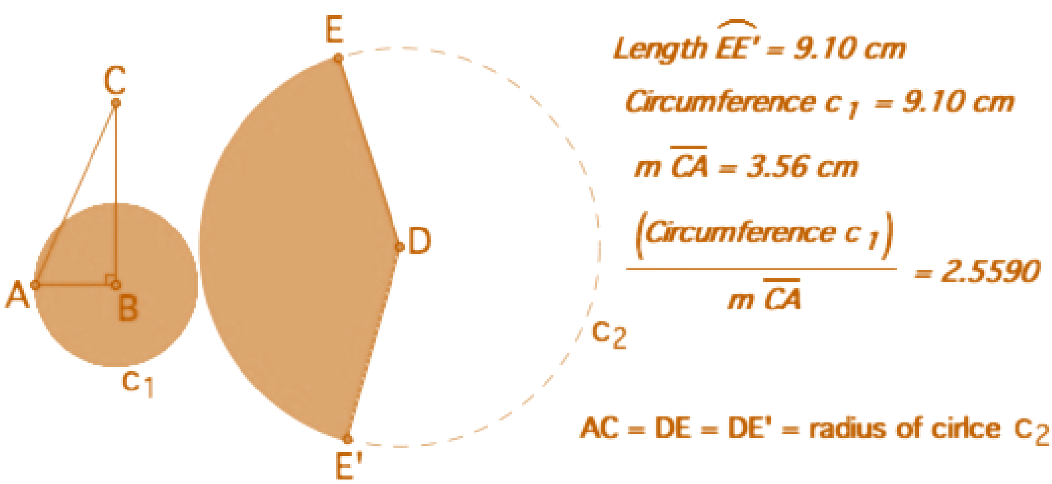

Figure 9. Cone net construction.

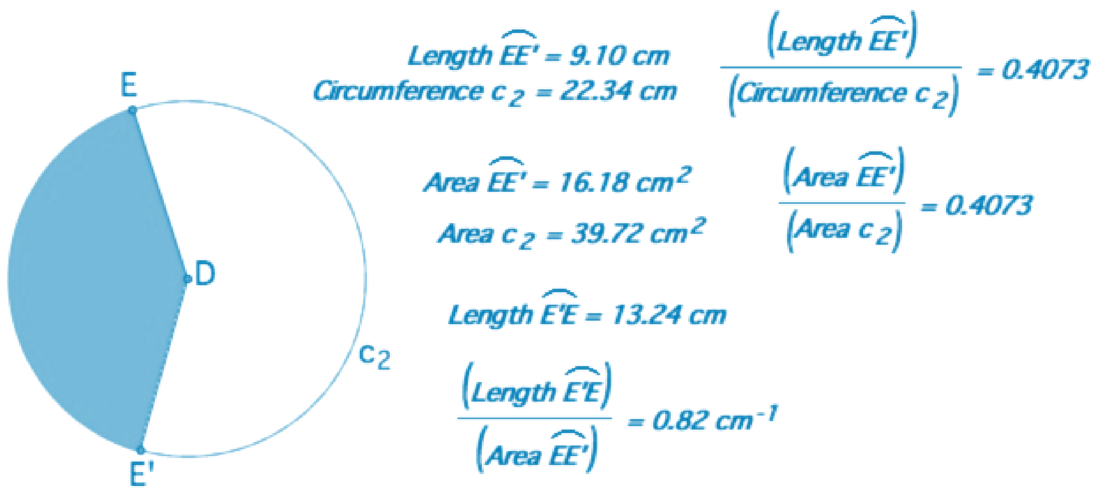

Figure 10. Proportional reasoning of cone net.

are equal and proportional to each other. I then asked the students if they could use this relationship to come up with the total surface area of the cone. I pointed out to them that: the total surface area of a cone is the area of the base (πr^2) + area of lateral surface.

The students stated the following:

- The base is a circle of radius r, and its circumference is equal to the arc length of the sector $EE'D$ (Figure 11). Since they know l, (let $l = DE = DE'$ = slant height of the cone) then they know the circumference of c_2.
- Since they know l — the radius of c_2 — then they know the area of c_2.

Since $\dfrac{\text{arc length } EE'}{\text{circumference } c_2}$ and $\dfrac{\text{area of sector}}{\text{area of circle } c_2}$ are proportional and the only part unknown is the lateral surface, which is the area of a sector:

$$\frac{\text{area of sector}}{\text{area of circle } c_2} = \frac{\text{arc length EE'}}{\text{circumference } c_2}$$

$$\frac{\text{area of sector}}{\pi l^2} = \frac{2\pi r}{2\pi l}$$

$$\text{area of sector} = \frac{2\pi r}{2\pi l} \times \pi l^2 = \pi r l$$

Therefore, total surface area of a cone = $\pi r^2 + \pi r l$

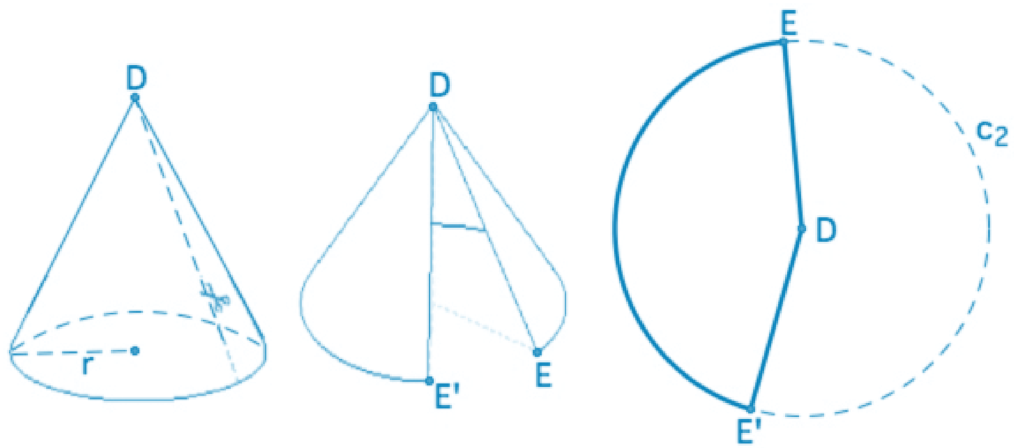

Figure 11. Lateral surface of a cone.

The students discussed the two approaches and noted that they now could see where the formulae for total surface area of a square based pyramid and a cone came from. At the end of this in-class investigation, students made the following additional comments:

> Having models with nets and also actually constructing them, leaves a mark in my memory for a long time. I now see how these nets are constructed using Geometer's Sketchpad — I will definitely use them with my students.
>
> The use of Geometer's Sketchpad made it easy to investigate the propositional relationship that enabled us to find the formula for calculating total surface area. With its dynamic nature, we were able to change the dimensions to help us build conjectures in terms of the relationships of parts in the constructed arcs, circles and areas."

Conclusion

These activities using models of pyramids and cones and their associated nets, in conjunction with technology, helped students understand the pertinent mathematics more conceptually. Giving students formulae is usually not enough for them to develop deep understanding. Teachers need to be encouraged to use visual objects in the form of models and nets to help students develop their spatial proportional reasoning. Some students may struggle when asked to find total surface areas, particularly if they have a difficult time visualising the unseen faces. Helping students to be able to derive the formula on their own is one way to help them succeed in finding and understanding total surface area of three dimensional solids.

References

Ainge, D. J. (1996). Upper primary students' constructing and exploring three dimensional shapes: A comparison of virtual reality with card nets. *Journal of Educational Computing Research, 14*(4), 345–369.

Bonotto, C. (2003). About students' understanding and learning the concept of surface area. In D. H. Clements & G. Bright (Eds), *Learning and teaching measurement.* Reston, VA: NCTM.

Jackiw, N. (2001). *The geometer's sketchpad 4.01* [computer software]. Berkeley, CA: Key Curriculum Press.

Liedtke, W. (1995). Developing spatial abilities in the early grades. *Teaching Children Mathematics, 2*, 12–18.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

From Helen Prochazka's

Scrapbook

In the Garden of Eden, God is giving Adam a geometry lesson and says to him: "Two parallel lines intersect at infinity. It can't be proved but I've been there."